An invitation to the calculus of variations

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September 24, 2025

Outline

1. What is the calculus of variations?
2. Some current projects
3. Food for thought

What is the calculus of variations?

Calculus of variations = studying the **minimization** of **functionals**

- **Functional**: a function whose input is a function and output is a number
  - $f \mapsto \int_\Omega f \, dx$ is a functional.
    - ★ Input: functions (which are integrable on Ω)
    - ★ Output: the integral of f over Ω

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- An example problem
  - Fix a domain Ω and a function $h : \partial\Omega \to \mathbb{R}$. (data)
  - Minimize

$$L[g] := \int_\Omega |\nabla g(x)|^2 \, dx \qquad \text{(functional)}$$

among

$$g : \Omega \to \mathbb{R} \text{ s.t. } g = h \text{ on } \partial\Omega \qquad \text{(constraints)}$$

Calculus vs. calculus of variations

Calculus

Minimize

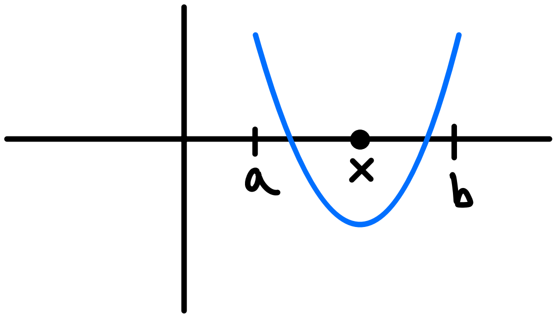
$f(x)$

among

x s.t. $a \leq x \leq b$.

To solve:

1. f continuous, $[a, b]$ closed and bounded $\implies$ min. exists
2. Find x s.t. $f'(x) = 0$
3. $f'' > 0$ on (a, b) $\implies$ sol. to step 2 is the minimizer



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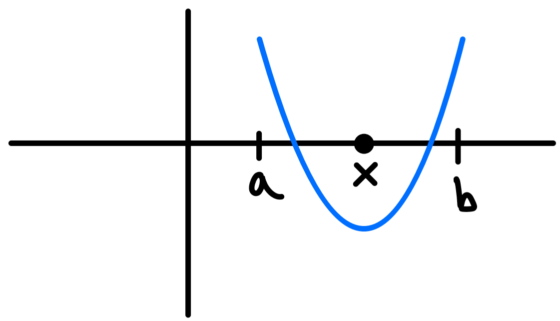

Calculus of variations

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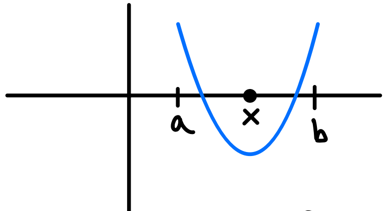

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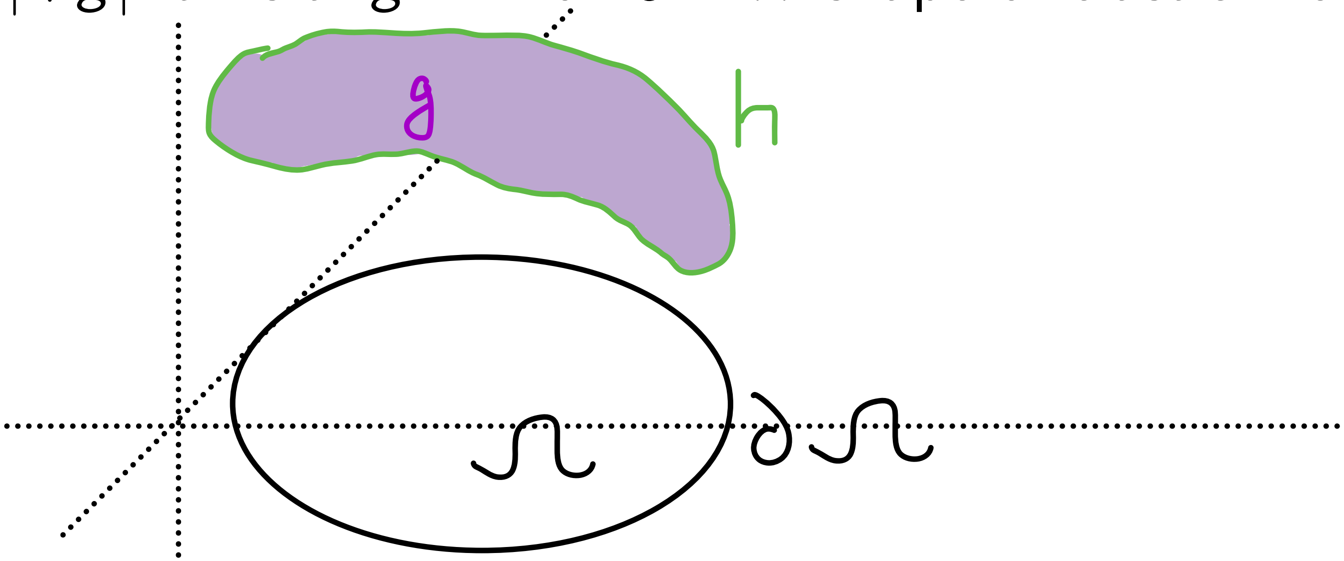
- Calculus in an infinite dimensional space is harder!
  - This caused a gap in Riemann's original proof of the Riemann mapping theorem.

Where do such problems arise?

Many physical systems exhibit "energy-minimizing" configurations.

Nature is lazy!

- Minimize $\int |\nabla g|^2 \, dx$ s.t. $g = h$ on $\partial\Omega \leftrightarrow$ shape of elastic membrane



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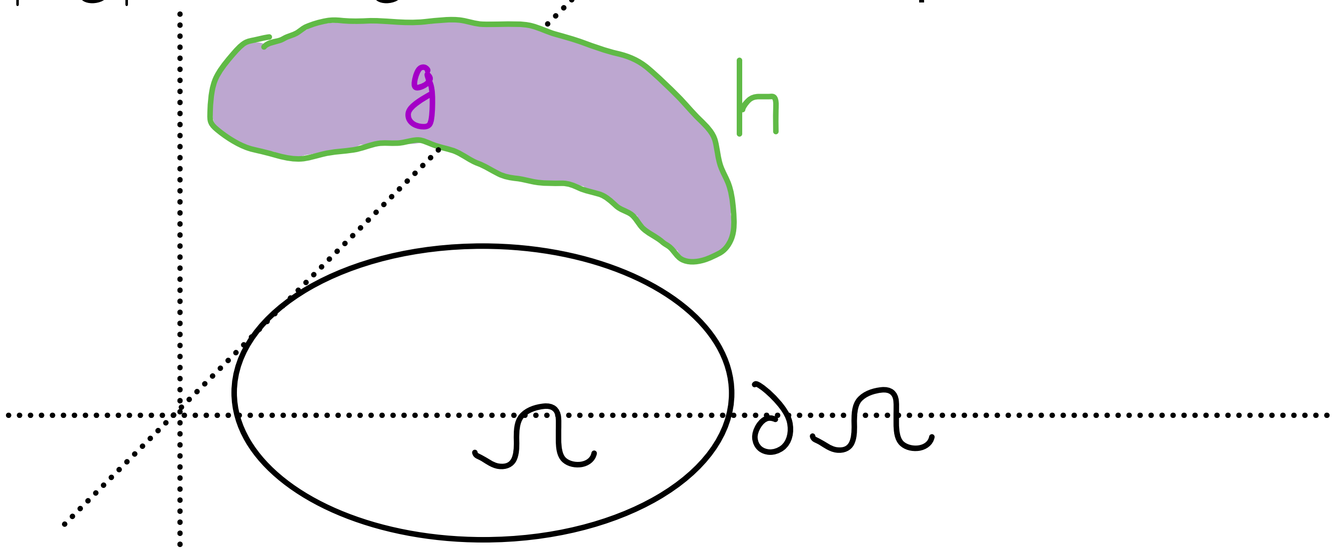

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- Calculus of variations also arises in: structural optimization, control theory, image processing, physics, economics, machine learning, and more. . .

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Mostly arising from materials science/physics. . .

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- Liquid crystals: flows $\sim$ liquid, crystalline structure $\sim$ solid
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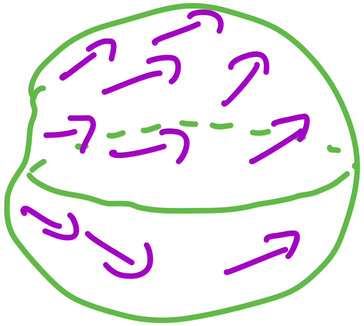
$$\text{Energy} = \int_{\partial\Omega} \sigma + |\nabla \vec{n}|^2 \, dS$$



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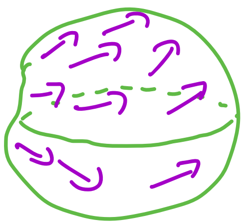
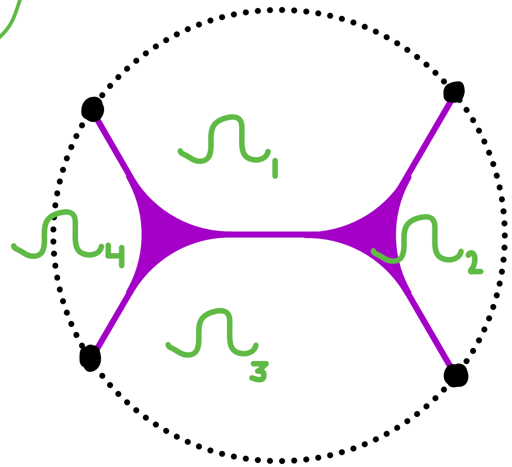


- Thin films
  - **Project:** Dynamics of foams

Energy $= \sum_i$ Surface areas of Ω_i



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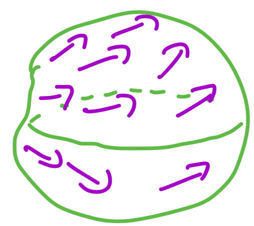

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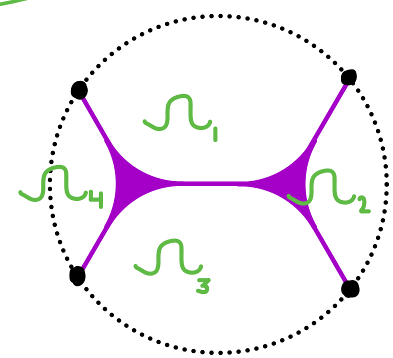
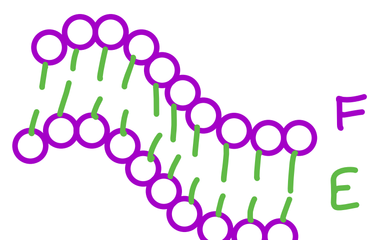


- Bilayer membranes
  - **Project:** Shape of minimizers



Energy $=$ Surface area of E $+$ “Transport cost” of F onto E

What kind of math is involved?

- **Analysis**
  - Partial differential equations
  - (Geometric) measure theory
- **What would you need to know/learn?**
- Class in measure theory e.g. 500
- Class in PDE's e.g. 512
- Reading course with me - e.g. first half of the book *Sets of finite perimeter and geometric variational problems*

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- Problems may require learning new things - e.g. interval arithmetic (rigorous computation with error estimates).

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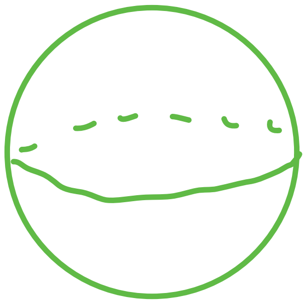
A closing example

- Gamow's liquid drop model for atomic nuclei
  - George Gamow (1920's) modelled protons/neutrons as an incompressible, uniformly charged fluid
  - Model explained nuclear fission in terms of a certain instability
  - Among sets E with fixed volume m, minimize

$$\text{Surface area of } \partial E + \frac{1}{8\pi}\int_E \int_E \frac{1}{|x-y|}\,dx\,dy$$



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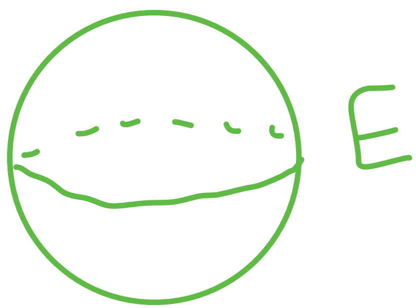

- What's known
  - Balls are unique minimizers if $0 < m < m_0$ (uses GMT)
  - Non-existence of minimizers for $m > m_1$

- What's unknown: does $m_0 = m_1$?

Conclusion

Read and come talk to me!

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  - Connection between calculus of variations and physics/mechanics
- *Energy driven pattern formation*
  - Calculus of variations as a paradigm in materials science
- *The regularity for the area functional in geometric measure theory*
  - History/current directions in GMT
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