An Indiscreet Invitation to Discrete Math: a network across mathematics

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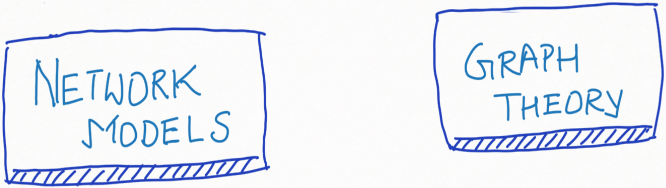
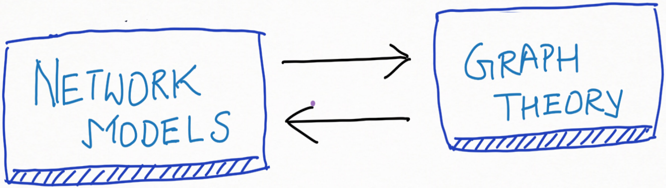
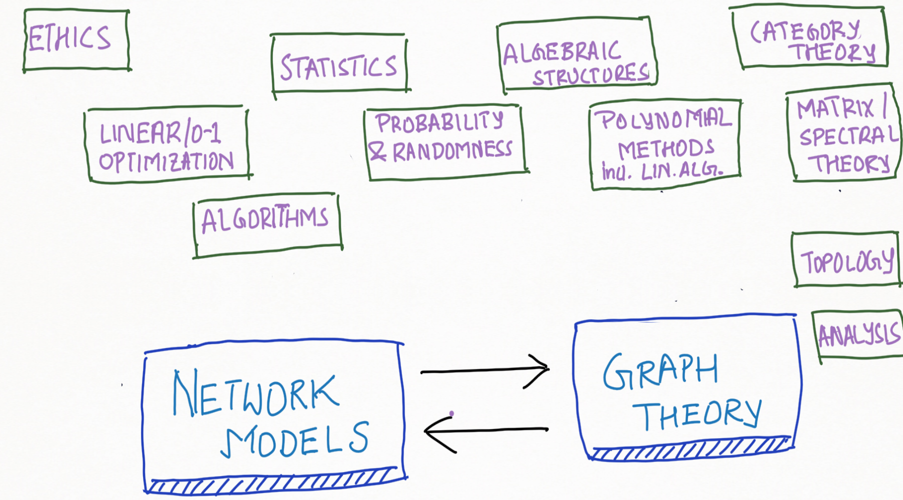
My view of Discrete Mathematics



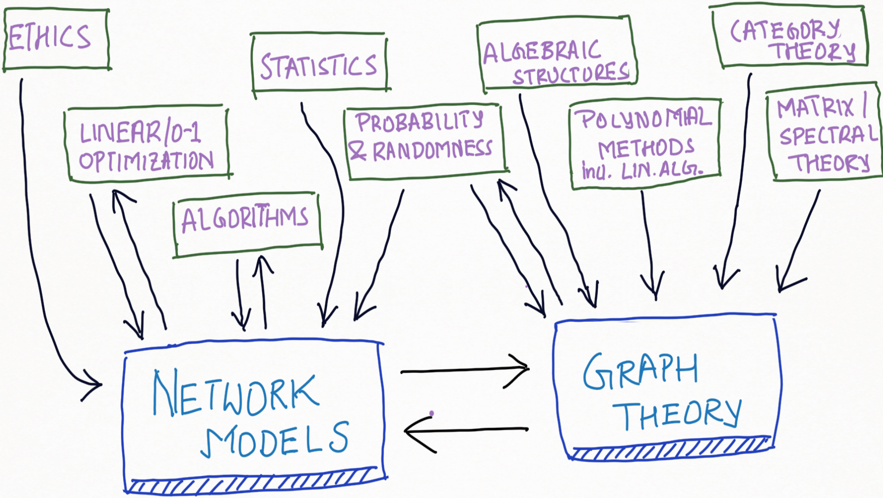
My view of Discrete Mathematics



My view of Discrete Mathematics



My work in Discrete Mathematics



Current Student Research

- Two recently defended PhD theses:

  Random and Fractional Perspectives On DP-coloring of Graphs by Daniel Dominik.

  On Spectral and Algorithmic Problems in Graph Theory by Bahar Kudarzi.
- Network design for equitable allocation of resources with Alaittin Kirtisoglu.
- List Coloring of Graphs with Leonardo Marciaga.
- Enumerative and Extremal problems on DP-coloring of graphs with Michelle Kang, and with Anne Ullyot.

Conflict-free Allocation of Scarce Resources

- Allocation of courses (limited resource) to timeslots (entities) so that courses with same instructor (conflict) are given different timeslots.
- Allocation of classrooms (limited resource) to courses (entities) so that courses with overlapping-time (conflict) are given different rooms.

Conflict-free Allocation of Scarce Resources

- Allocation of classrooms (limited resource) to courses (entities) so that courses with overlapping-time (conflict) are given different rooms.
- Allocation of radio channels (limited resource) to radio stations (entities) so that stations with proximity interference (conflict) are given different channels.
- Allocation of colors (limited resource) to regions (entities) in a map so that regions with common boundary (conflict) are given different colors.

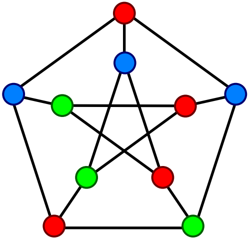
Such optimization problems are studied as "coloring" problems in Graph Theory, the mathematical theory of structures underlying networks.

Coloring a Graph

- Entities $\leftrightarrow$ Vertices.
  Conflicts $\leftrightarrow$ Edges.

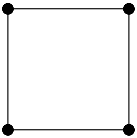
- Color vertices so that any vertices with an edge between them must get different colors.

- Resources $\leftrightarrow$ Colors.

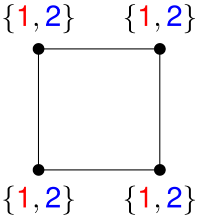


Given a graph *G* and a number *k*, we want to know if there is at least one such coloring of *G* using upto *k* colors.

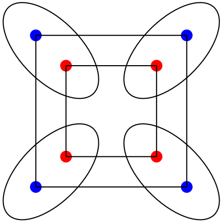
A More General Perspective



Graph G



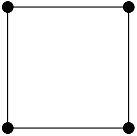
Colors for G



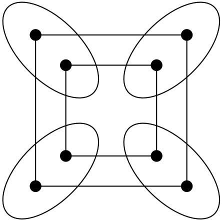
Cover for G

In the cover of G, vertices correspond to the available colors for G, and edges correspond to conflicts between those colors based on edges of G.

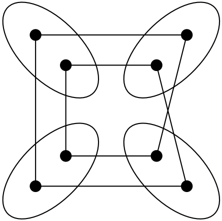
A More General Perspective



Graph G



A Cover for G



Another Cover for G

In the cover of G, vertices correspond to the available colors for G, and edges correspond to conflicts between those colors based on edges of G.

A More General Perspective

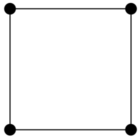
A topological aside:

What we are informally calling cover of a graph, can be formally defined in the language of covering map. A graph is a topological space, a one-dimensional simplicial complex, and covering maps can be defined and studied for graphs.

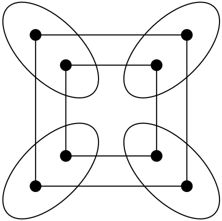
A surjective map $\phi : V(H) \to V(G)$ where G, H are graphs is a covering map if for every $x \in V(H)$, the neighbor set $N_H(x)$ is mapped bijectively to $N_G(\phi(x))$. When such a mapping exists and is k-to-1, we say that H is a k-lift, or k-fold cover of G.

Lifts of graphs have been studied in algebraic/ topological graph theory since 1980s (see Godsil & Royle, Algebraic Graph Thry); and in random graph theory since 2000 (see seminal papers of Linial).

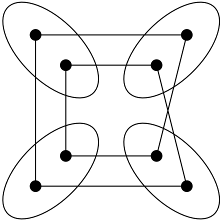
A More General Perspective



Graph *G*



A Cover for *G*



Another Cover for *G*

A cover of *G* can be expressed with a <u>permutation</u> on each edge of *G*. The permutation models the conflict betwwen those colors (resources).

S-labeled Graphs and Coloring

Jin, Wang, Zhu (2019) (although these ideas go back to 1990s in topology and topological graph theory):

- Let A be a finite set of colors, $|A| = k$, and $S \subseteq S_A$ be a subset of the permutations of A.
- An S-labeling of G is a pair (D, σ) consisting of an orientation D of G and an edge labeling $\sigma : E(D) \to S$.
- An S-k-coloring of (D, σ) is $\kappa : V(G) \to A$ such that for each edge $(u, v) \in E(D)$ if $\pi = \sigma(u, v)$ then $\pi(\kappa(u)) \neq \kappa(v)$.

A Poset of (notions of) Graph Colorings

- Coloring of S-labeled graphs is a common generalization of many well studied notions of colorings.

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- Coloring of S-labeled graphs is a common generalization of many well studied notions of colorings.
- $S = \{\mathrm{id}_A\}$ gives classical coloring. Introduced in 1850s.
- $S = \mathcal{L}$, linear permutations, gives Signed-coloring. Introduced in 1930s with many applications in context of psychological models, root systems, Ising model, etc.
- Signed $\mathbb{Z}_k$-coloring.
- Group $\mathbb{Z}_k$-coloring.
- Coloring of Gain graphs.
- $S = S_A$ gives DP-coloring. Introduced in 2015 and widely studied since then.

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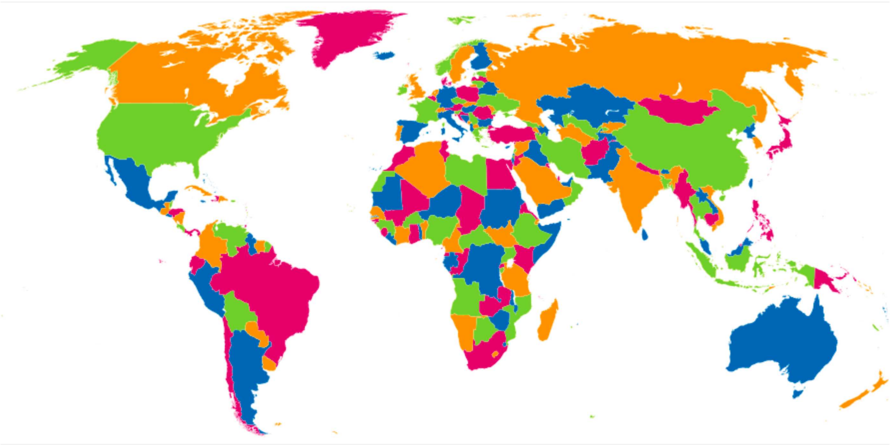
- Any choice of subset of permutations $S \subseteq S_A$ leads to a notion of coloring.

- The subset relation over the symmetric group, induces a partial order on all these notions of coloring with the DP coloring as the unique maximal element and the classical coloring as a minimal element.
  In fact, it's a distributive lattice of notions of colorings.

Four Colors for the World Map



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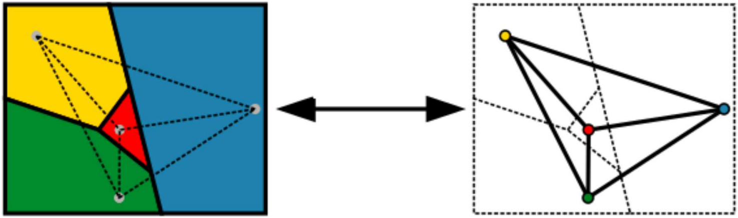
Coloring a Planar Graph



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The Origins of Graph Coloring

- Francis Guthrie (October 23, 1852): Four colors suffice for any planar graph?
- Frederick Guthrie (1852): Asked his professor Augustus De Morgan.
- De Morgan (1852) enquired with William Hamilton.
- Arthur Cayley presented it to the London Mathematical Society (LMS).
- Kempe (1879) published a proof claiming to solve it. Honored as Fellow of the Royal Society and elected President of LMS.
- Heawood (1890) found an error in Kempe's proof. The fixed proof showed: Five colors suffice for any planar graph.

Counting the number of colorings

- Birkhoff (1912): Chromatic Polynomial, $P(G, k)$, the number of colorings of G using k colors.
- Four Color Conjecture (1852): $P(G, 4) > 0$ for every planar graph G.
- Five Color Theorem (Kempe (1879), Heawood (1890)): $P(G, 5) > 0$ for every planar graph G.

- Birkhoff and Lewis (1946) conjectured:
  For any planar graph G,
  $P(G, k) \geq k(k-1)(k-2)(k-3)^{n-3}$ for all real numbers $k \geq 4$.

  They proved this is true for $k \geq 5$, thus giving exponentially many 5-colorings of planar graphs: $P(G, 5) > 2^n$.

More progress and more questions

- Grötzsch (1959): $P(G,3) > 0$, for any triangle-free planar graph.

- Appel and Haken (1976): Four Color Theorem! $P(G,4) > 0$ for every planar graph G.

- Vizing (1975), Erdös, Rubin, and Taylor (1979): Introduced List Coloring. Instead of same colors for each vertex, vertices are assigned lists of (possibly different) colors. Kostochka and Sidorenko (1990): List Color Function, $P_\ell(G,k)$, the guaranteed number of list colorings of G, no matter which lists of k-colors are available for each vertex.
- $P_\ell(G,k) \leq P(G,k)$.

- Thomassen (1995): $P_\ell(G,5) > 0$ for any planar graph G.

Exponentially Many Colorings of Planar Graphs!

The history of coloring of planar graphs and its subfamilies, is intertwined with the related conjectures on existence of exponentially many such colorings going back at the least to Birkhoff's and Whitney's works in 1930s.

Exponentially Many Colorings of Planar Graphs!

- Birkhoff and Lewis (1946): $P(G, 5) > 2^n$.
- Thomassen (2007): $P_\ell(G, 5) > 2^{n/9}$.

- Since 1990s, there has been much work done on showing that planar graphs and their subfamilies have exponentially many list k-colorings for appropriate $k \in \{3, 4, 5\}$.

- These proofs are typically intricate topological arguments specialized to the family of planar graphs under consideration.

Can we unify these results and arguments in a systematic way?

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- For $S \subseteq S' \subseteq S_A$, $P_S(G,k) \leq P_{S'}(G,k)$.

- $P_{DP}(G,k) \leq \ldots \leq P_{\mathcal{L}}(G,k) \leq \cdots \leq P(G,k)$; and $P_{DP}(G,k) \leq \ldots \leq P_{\ell}(G,k) \leq \cdots \leq P(G,k)$.
- Exponential lower bound on $P_{DP}(G,k)$ would give an exponential lower bound on all these (and more) colorings of G.

Exponentially Many Colorings

Theorem (Dahlberg, K., Mudrock (2024+))

Let $k = p^r$ where p is prime, $r \in \mathbb{N}$, and $k > 2$.
Suppose G is a connected n-vertex simple graph with m edges.
If $P_{DP}(G, k) > 0$ and $m \leq 2n - \frac{k-3}{k-2}$, then

$$P_{DP}(G, k) \geq k^{((2n-m)(k-2)-(k-3))/(k-1)}.$$

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Suppose G is a connected n-vertex simple graph with m edges.
If $P_{\mathcal{L}}(G, k) > 0$ and $m \leq (k-1)n$, then

$$P_{\mathcal{L}}(G, k) \geq k^{n - \frac{m}{k-1}}.$$

Polynomial Method

Terrence Tao describes the polynomial method as:

"strategy is to capture the arbitrary set of objects in the zero set of a polynomial whose degree is in control; for instance the degree may be bounded by a function of the number of the objects."

Then we use algebraic tools to understand this zero set.

This paradigm has been used for breakthrough results in arithmetic combinatorics, additive combinatorics, number theory, graph theory, discrete geometry, and more.

Polynomial Method for Exponentially Many Colorings

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We define a polynomial over a finite field such that its non-roots correspond to proper colorings of G. Then we study the roots/non-roots of this polynomial over a discretized subset of the field.

A Unification of Theorems and Methods

Theorem (Dahlberg, K., Mudrock (2024+))

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Application of our lower bounds to families of planar graphs either improve previously known results or are the first such known results.

Moreover, this gives a simple-to-apply unified perspective, both in terms of the statement and the underlying method, for many notions of colorings applied to sparse graphs without requiring any topological assumptions.

An Example of Our Applications

Theorem (Dahlberg, K., Mudrock (2024+))

Let G be an n-vertex planar graph, and k be a power of prime.

1. *If G has no cycle of length in $\{4,5,6,7,8\}$, then $P_{DP}(G,k) \geq k^{\frac{n}{5}\frac{k-2}{k-1}-1}$ for $k \geq \chi_{DP}(G)$.*
2. *If G has no cycle of length in $\{4,5,6,9\}$, then $P_{DP}(G,k) \geq k^{\frac{n}{11}\frac{k-2}{k-1}-1}$ for $k \geq 3$. In particular, $P_{DP}(G,3) \geq 3^{\frac{n}{22}-1}$.*
3. *If G has no intersecting triangles and no cycle of length in $\{4,5,6,7\}$, then $P_{DP}(G,k) \geq k^{\frac{2n}{13}\frac{k-2}{k-1}-1}$ for $k \geq 3$. In particular, $P_{DP}(G,3) \geq 3^{\frac{n}{13}-1}$.*
4. *If G has no cycle of length in $\{4,5,6\}$, then $P_{DP}(G,k) \geq k^{\frac{n}{11}\frac{k-2}{k-1}-1}$ for $k \geq 4$. In particular, $P_{DP}(G,4) \geq 3^{\frac{n}{33}-1}$.*
5. *If G has no cycle of length in $\{4,5,7,9\}$, then $P_{DP}(G,k) \geq k^{\frac{2n}{13}\frac{k-2}{k-1}-1}$ for $k \geq 3$. In particular, $P_{DP}(G,3) \geq 3^{\frac{n}{22}-1}$.*

What would you like to work on?

Come join the fun!